Those wonderful elastic waves

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Abstract

We consider in a simple and general way elastic waves in isotropic and anisotropic media, their polarization, speeds, reflection from interfaces with mode conversion, and surface waves. Reflection of quasi transverse waves in anisotropic media from a free surface is shown to be characterized by three critical angles.

I. INTRODUCTION

In our time of supercollider, quantum computing, teleportation, dark matter and an eager search for a new physics, acoustics and elastic waves look like an archaic science. Am.J.Phys. rarely publish paper on this topic. In fact, we have found the single article of 1980 [1] relevant to our consideration. Indeed this science looks archaic, because everything seems to be well resolved and the science became like an engineering tool, frequently used in many applications. We will show that this impression is wrong.

The theory seems to be well established (see, for instance [2]), and all the textbooks [3], are unanimous in its presentation. The main notion is a displacement vector $\boldsymbol{u}(\boldsymbol{r},t)$ of a material point at a position $\boldsymbol{r}$ at a time moment t. Its Descartes components $u_i(\boldsymbol{r},t)$ obey the Newtonian equation of motion

$$\rho\frac{\partial^2}{\partial t^2}u_i(\boldsymbol{r},t)=\frac{\partial}{\partial x_j}\sigma_{ij}(\boldsymbol{r},t), \tag{1}$$

where ρ is the material density, x_j are components of the radius-vector $\boldsymbol{r}$, σ_{ij} is a stress tensor

$$\sigma_{ij}=c_{ijkl}u_{kl}, \tag{2}$$

which is proportional to the deformation tensor u_{ij}

$$u_{jk}=\frac{1}{2}\left(\frac{\partial u_j}{\partial x_k}+\frac{\partial u_k}{\partial x_j}\right), \tag{3}$$

and coefficients of proportionality c_{ijkl} in (2) comprise themselves a tensor with the symmetries:

$$c_{ijkl}=c_{jikl}=c_{ijlk}=c_{klij}. \tag{4}$$

As is usual, in (1), (2) and everywhere below a summation over repeated indices is assumed.

In isotropic media the tensor c_{ijkl} is very simple:

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{kj} + \delta_{ik}\delta_{lj}), \tag{5}$$

where δ_{ij} is the Kronecker symbol equal to unity for $i = j$ and to zero otherwise, and λ, μ are two parameters called Lamé elastic constants.

In the case of anisotropic media (usually crystals) the tensor c_{ijkl} contains considerably more parameters [4–7]. They are phenomenological, and their physical meaning is not sufficiently clear.

The displacement vector $\boldsymbol{u}(\boldsymbol{r}, t)$ is usually represented as a sum $\boldsymbol{u}(\boldsymbol{r}, t) = \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\times\boldsymbol{\psi}$ of two parts, where φ is a scalar, $\boldsymbol{\psi}$ is a vector potentials, $\boldsymbol{\nabla}$ is the differential vector

$$\boldsymbol{\nabla} = \boldsymbol{e}_x\nabla_x + \boldsymbol{e}_y\nabla_y + \boldsymbol{e}_z\nabla_z \equiv \boldsymbol{e}_x\frac{\partial}{\partial x} + \boldsymbol{e}_y\frac{\partial}{\partial y} + \boldsymbol{e}_z\frac{\partial}{\partial z}, \tag{6}$$

and $\boldsymbol{e}_{x,y,z}$ are unit vectors along three axes of an orthogonal Descartes reference frame.

We will show that the use of the scalar and vector potentials is not necessary. They only complicate the theory. Everything can be presented in much more simple and transparent form with a wave function, like in particle physics. In the case of isotropic media such a presentation makes the theory looking almost trivial.

In the case of anisotropic media it is quite instructive to consider not crystals, but a medium with an anisotropy distinguished only by a single direction [8], because a single vector is sufficient to elucidate the difference of isotropic and anisotropic media. We found very interesting counter intuitive features of waves reflected from interfaces in anisotropic media. Surprisingly, but these features have not yet been discussed in literature.

After general introduction in the second section to the theory of elastic waves in isotropic and anisotropic media, we in the third section consider isotropic ones. In isotropic media elastic waves are very naturally break up into three classes, which are called modes: two modes with transverse and one mode with longitudinal polarizations.

Theory of waves in a homogeneous medium is rather primitive. It becomes more rich when the medium contains an interface or a free surface. In that case a wave reflects, refracts and multiply splits at the interface. The presence of a surface or an interface gives rise also to waves of the fourth mode — the surface waves, which run along the surface, exponentially decay away from it, and have a mixed polarization. We consider how do they appear and what are their properties.

In the fourth section we go to anisotropic media with a single anisotropy vector. There again all waves break up into three classes-modes, but only one mode has purely transverse polarization. The other two are hybrids which are nor transverse nor longitudinal. One of these modes is called quasi transverse and the other one — quasi longitudinal, because the smaller is the anisotropy parameter, the closer is their polarization to pure transverse and longitudinal directions respectively.

This is more or less evident and simple. The complications start when a wave meets an interface or a free surface. In that case we have reflection and refraction, which are in general accompanied with triple waves splitting. Specular reflection is absent, and surface waves become exotic. An experience acquired from isotropic media considerations leads to a conclusion that in anisotropic ones an incident wave can completely transform into a surface wave! It is absolutely unacceptable because of the energy conservation law. The incident wave carries an energy, which must accumulate in the surface one, therefore the amplitude of the last one must grow exponentially. It is impossible to describe such a process by a linear stationary wave theory. So we meet a paradox: the theory predicts some non physical solutions, which cannot be described by the theory.

We considered the problem in details, and found that the reflected waves do not accumulate into a surface mode because of unexpected counter intuitive properties of the elastic waves in anisotropic media. For instance, besides the first critical grazing angle φ_c, similar to that one in isotropic media, at which a quasi longitudinal wave becomes of the surface type, there is a second critical angle, $\varphi_{c1} < \varphi_c$, when the reflected wave, which is naturally thought of as moving away from the reflecting surface, changes its direction, as if starting to move toward it, though its energy flux remains going away from the surface.

There is also a third critical angle, $\varphi_{c2} < \varphi_{c1}$, at which the energy flux toward the surface becomes zero. This angle is not zero, i.e. $\varphi_{c2} > 0$, and annulation of energy flux at this angle means that we cannot direct a ray of elastic waves to the surface at an angle $\varphi < \varphi_{c2}$. It is really strange, but we consider it as a hint, which the linear theory of elasticity gives us to point out the cases, where we have to involve a nonlinearity. We could not understand how to introduce it, but we were so much impressed by the unexpected properties of the elastic waves, that we decided to relate about them to the readers of this journal.

Our research was started because of a need to explain the difference between theoretically predicted and experimentally measured anisotropy of sound speed in rocks [8]. The theory, based on texture of rocks measured at a neutron diffractometer, predicted

anisotropy of speeds in some rocks three times lower than the one measured with ultrasound. We hope that our results will shed light on this difference, and will be checked in an experiment of the kind discussed in [1].

II. THE MAIN EQUATIONS FOR ELASTIC WAVES

The starting point for study of elasticity is the free energy density of a medium deformation [2]. For isotropic media it is

$$F = \frac{\lambda}{2} u_{ll}^2 + \mu u_{lj}^2, \tag{7}$$

where u_{ij} is the deformation tensor (3), λ, μ are the Lamé elastic constants, notation u_{lj}^2 means $u_{lj}u_{jl}$, and summation over repeated indices is assumed.

In anisotropic case we have to distinguish a direction, say along a unit vector $\boldsymbol{a}$, and introduce a new elastic constant, say ζ, which is of the same dimension of energy density as λ and μ. Then the free energy becomes

$$F = \frac{\lambda}{2} u_{ll}^2 + \mu u_{lj}^2 - \zeta[(a_j u_{jl})^2 + (u_{jl} a_l)^2], \tag{8}$$

where a_j are Descart components of the vector $\boldsymbol{a}$, anisotropic part is proportional to square of the vector with components $a_j u_{ji}$, and the sign before ζ is not necessary negative. Though the two anisotropic terms in square brackets are identical (u_{lj} is symmetrical) we put them separately to get the tensor c_{ijkl} with required symmetry (4).

The anisotropy term shows that the energy of deformation depends not only on change of volume (the term $\propto \lambda$) and change of shape (the term $\propto \mu$), but also on angles of both deformations with respect to the anisotropy vector $\boldsymbol{a}$ (the term $\propto \zeta$).

A question can be raised here: why do we use such an anisotropy modification of the isotropic free energy (7)? Is it not possible to find a different one? The reply is yes. It is possible to use a different modification. For instance, instead of (8) we can accept the free energy in the form

$$F = \frac{\lambda}{2} u_{ll}^2 + \mu u_{lj}^2 - \zeta (a_j u_{jl} a_l)^2, \tag{9}$$

or we can use a combinations of anisotropic terms in (9) and in (8). We chose (8), and it is an arbitrariness. We think that consideration of different types of anisotropy is a good task for students.

With the free energy we can define the stress tensor

$$\sigma_{ij} = \frac{\partial F}{\partial u_{ij}}, \tag{10}$$

which after substitution of (8) gives

$$\sigma_{ij} = \lambda\delta_{ij}u_{ll} + 2\mu u_{ij} - 2\zeta(a_i u_{jl} a_l + a_l u_{li} a_j) = c_{ijkl}u_{kl}, \tag{11}$$

where

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{kj} + \delta_{ik}\delta_{lj}) - \zeta(a_i\delta_{jl}a_k + \delta_{il}a_j a_k + a_i\delta_{jk}a_l + \delta_{ik}a_j a_l). \tag{12}$$

The expression for isotropic medium is obtained in the limit $\zeta \to 0$. Note that the tensor c_{ijkl} satisfies the symmetry requirements (4). In the case of (9) the anisotropic part of the tensor c_{ijkl} will be simply a product $\zeta a_i a_j a_k a_l$.

With the stress tensor (11) the Newtonian equation of motion (1) for the displacement vector becomes

$$\rho\ddot{u}_i = \nabla_j\sigma_{ij} = \mu[\Delta u_i + \nabla_i(\nabla\cdot\boldsymbol{u})] + \lambda\nabla_i(\boldsymbol{\nabla}\cdot\boldsymbol{u}) -$$

$$-\zeta\left(a_i[\Delta(\boldsymbol{u}\cdot\boldsymbol{a}) + (\boldsymbol{a}\cdot\boldsymbol{\nabla})(\boldsymbol{\nabla}\cdot\boldsymbol{u})] + (\boldsymbol{a}\cdot\boldsymbol{\nabla})^2 u_i + \nabla_i(\boldsymbol{a}\cdot\boldsymbol{\nabla})(\boldsymbol{a}\cdot\boldsymbol{u})\right), \tag{13}$$

where ρ is the medium density.

We can seek solution of (13) in the form of a complex plain wave $\boldsymbol{u}(\boldsymbol{r},t) = \boldsymbol{A}\exp(i\boldsymbol{kr} - i\omega t)$, like a wave function in particle physics, where vector $\boldsymbol{A}$ ia a unit polarization vector. Of course, the elastic waves are real waves, so they are represented by the real part of the complex wave function. Later we shall see, where we must be especially careful in description of elastic waves with complex function, but for now we meet no difficulties, and after substitution of such $\boldsymbol{u}$ into (13) we obtain an equation for $\boldsymbol{A}$:

$$\rho\omega^2\boldsymbol{A} = \mu k^2\boldsymbol{A} + (\lambda+\mu)\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{A}) -$$

$$-\zeta\left(\boldsymbol{a}[k^2(\boldsymbol{a}\cdot\boldsymbol{A}) + (\boldsymbol{k}\cdot\boldsymbol{a})(\boldsymbol{k}\cdot\boldsymbol{A})] + (\boldsymbol{k}\cdot\boldsymbol{a})[(\boldsymbol{k}\cdot\boldsymbol{a})\boldsymbol{A} + \boldsymbol{k}(\boldsymbol{a}\cdot\boldsymbol{A})]\right). \tag{14}$$

It is convenient to transform this equation to dimensionless form dividing both parts of (14) by μk^2. After introduction of the standard transverse speed $c_t = \sqrt{\mu/\rho}$, the phase speed of the wave $V = \omega/k$, dimensionless ratio $v = V/c_t$, dimensionless parameters $E = (\lambda+\mu)/\mu$, $\xi = \zeta/\mu$ and the unit vector $\boldsymbol{\kappa} = \boldsymbol{k}/k$ the equation becomes

$$\Omega^2\boldsymbol{A} = E\boldsymbol{\kappa}(\boldsymbol{\kappa}\cdot\boldsymbol{A}) - \xi\Big[\{\boldsymbol{a}(\boldsymbol{a}\cdot\boldsymbol{A}) + (\boldsymbol{\kappa}\cdot\boldsymbol{a})\Big[\boldsymbol{a}(\boldsymbol{\kappa}\cdot\boldsymbol{A}) + \boldsymbol{A}(\boldsymbol{a}\cdot\boldsymbol{\kappa}) + \boldsymbol{\kappa}(\boldsymbol{A}\cdot\boldsymbol{a})\Big]\Big\}, \tag{15}$$

where $\Omega^2 = v^2 - 1$. First we put $\xi = 0$, and consider an isotropic medium.

III. WAVES IN ISOTROPIC MEDIA ($\xi = 0$)

For isotropic media the equations (13-15) are reduced respectively to

$$\rho \ddot{u}_i = \nabla_j u_{ij} = \mu[\Delta u_i + \nabla_i(\nabla \cdot \boldsymbol{u})] + \lambda \nabla_i(\boldsymbol{\nabla} \cdot \boldsymbol{u}), \tag{16}$$

$$\rho \omega^2 \boldsymbol{A} = \mu k^2 \boldsymbol{A} + (\lambda + \mu)\boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{A}), \tag{17}$$

$$\Omega^2 \boldsymbol{A} = E\boldsymbol{\kappa}(\boldsymbol{\kappa} \cdot \boldsymbol{A}). \tag{18}$$

For the given propagation direction $\boldsymbol{\kappa}$ we can introduce two orthonormal vectors $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$, which are perpendicular to $\boldsymbol{\kappa}$. In the orthonormal basis $\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)}$, $\boldsymbol{\kappa}$ the polarization unit vector $\boldsymbol{A}$ is representable as

$$\boldsymbol{A} = \alpha^{(1)}\boldsymbol{e}^{(1)} + \alpha^{(2)}\boldsymbol{e}^{(2)} + \beta\boldsymbol{\kappa}. \tag{19}$$

After multiplication of (18) consecutively by $\boldsymbol{e}^{(1,2)}$ and $\boldsymbol{\kappa}$ we obtain three equations for coordinates $\alpha^{(1,2)}$ and β:

$$\Omega^2 \alpha^{(1,2)} = 0, \qquad [\Omega^2 - E]\beta = 0. \tag{20}$$

They are independent and give three solutions

$$\boldsymbol{A}^{(1,2)} = \boldsymbol{e}^{(1,2)}, \qquad \boldsymbol{A}^{(3)} = \boldsymbol{\kappa}. \tag{21}$$

Their speeds are determined by equations

$$(v^{(1,2)})^2 = 1 \to V^{(1,2)} = \sqrt{\frac{\mu}{\rho}} = c_t, \qquad (v^{(3)})^2 = E+1 \to V^{(3)} = \sqrt{E+1}c_t = \sqrt{\frac{\lambda + 2\mu}{\rho}} = c_l. \tag{22}$$

Since $V = \omega/k$, we can tell that for a given frequency ω the wave numbers $k = |\boldsymbol{k}|$ of three modes are

$$k^{(1,2)} = \frac{\omega}{c_t}, \qquad k^{(3)} = \frac{\omega}{c_l}. \tag{23}$$

The two unit vectors $\boldsymbol{e}^{(1,2)}$ above are orthogonal to each other, and lie in the plane, perpendicular to the unit vector $\boldsymbol{\kappa}$, but their azimuthal angle around $\boldsymbol{\kappa}$ can be arbitrary. We can use this freedom to facilitate solution of different problems. In particular, below, when we consider reflection from an interface. There we can choose $\boldsymbol{e}^{(1)}$ to be perpendicular to the incidence plane, and $\boldsymbol{e}^{(2)}$ to be inside it.

A. Reflection from an interface

Suppose that the medium consists of two parts: one at $z<0$ with constants λ, μ, ρ and another one at $z>0$ with constants λ', μ', ρ', then there is reflection and refraction of waves at the interface $z=0$. If a plane wave $\boldsymbol{u}_{inc}(\boldsymbol{r},t)=\boldsymbol{A}\exp(i\boldsymbol{k}\boldsymbol{r}-i\omega t)$ incident from $z<0$ is of mode $\boldsymbol{A}^{(j)}$ (j is one of the numbers 1, 2 or 3) then the interface transforms this displacement vector to

$$\boldsymbol{u}(\boldsymbol{r},t)=\exp(i\boldsymbol{k}_{\parallel}\boldsymbol{r}_{\parallel}-i\omega t)\times$$
$$\times\left[\left(\boldsymbol{A}^{(j)}e^{ik_{\perp}^{(j)}z}+\sum_{l=1}^{3}r^{(jl)}\boldsymbol{A}_R^{(l)}e^{-ik_{\perp}^{(l)}z}\right)\Theta(z<0)+\sum_{l=1}^{3}t^{(jl)}\boldsymbol{A}_T^{(l)}e^{ik_{\perp}^{\prime(l)}z}\Theta(z>0)\right], \qquad (24)$$

where $r^{(jl)}$, $t^{(jl)}$ are reflection and refraction amplitudes of the l-th mode ($l=1,2,3$) for the incident j-th mode, and Θ is a step function, which is equal to unity, when inequality in its argument is satisfied, and to zero otherwise.

Below the interface ($z<0$) the displacement consists of the incident wave of j-th mode and reflected waves of modes $\boldsymbol{A}_R^{(l)}$ (the lower index R means reflected). Above the interface ($z>0$) the displacement consists of transmitted waves of modes $\boldsymbol{A}_T^{(l)}$ (the lower index T means transmitted).

All the waves differ from one another not only by polarization, but also by the wave vector $\boldsymbol{k}$, which can be represented as

$$\boldsymbol{k}=\boldsymbol{k}_{\parallel}+\boldsymbol{k}_{\perp}\equiv\boldsymbol{\tau}k_{\parallel}\pm\boldsymbol{n}k_{\perp}, \qquad (25)$$

where $\boldsymbol{\tau}$ is a unit vector along the interface, and $\boldsymbol{n}$ is a unit vector along z-axis perpendicular to the interface, as is shown in Fig. 1. Note, that the component, $\boldsymbol{k}_{\parallel}=\boldsymbol{\tau}k_{\parallel}$, of the wave vectors is identical for all the waves, as is demonstrated by the first common factor $\exp(i\boldsymbol{k}_{\parallel}\boldsymbol{r}_{\parallel}-i\omega t)$ in (24), where $\boldsymbol{r}_{\parallel}$ are coordinates in the interface. The vector $\boldsymbol{k}_{\parallel}$ is identical, because the space along $\boldsymbol{\tau}$ is uniform and nothing can change this component.

The normal components $k_{\perp}^{(l)}$ and $k_{\perp}^{\prime(l)}$ of wave vectors of modes $\boldsymbol{A}_R^{(l)}$ and $\boldsymbol{A}_T^{(l)}$ respectively, are positive numbers and their value depends on the mode l. Since $k_{\perp}^2=k^2-k_{\parallel}^2$, then from (23) it follows that

$$k_{\perp}^{(1,2)}=\sqrt{\frac{\omega^2}{c_t^2}-k_{\parallel}^2},\quad k_{\perp}^{\prime(1,2)}=\sqrt{\frac{\omega^2}{c_t^{\prime 2}}-k_{\parallel}^2},\quad k_{\perp}^{(3)}=\sqrt{\frac{\omega^2}{c_l^2}-k_{\parallel}^2},\quad k_{\perp}^{\prime(3)}=\sqrt{\frac{\omega^2}{c_l^{\prime 2}}-k_{\parallel}^2}, \qquad (26)$$

where $c_{t,l}$, and $c'_{t,l}$, are the speeds defined in (22) for lower and upper spaces respectively.

To find reflection and refraction amplitudes, we need boundary conditions. One of them is continuity of the displacement vector:

$$\boldsymbol{u}|_{z=-0} = \boldsymbol{u}|_{z=+0} \to \boldsymbol{A}^{(j)} + \sum_{l=1}^{3} r^{(jl)} \boldsymbol{A}_R^{(l)} = \sum_{l=1}^{3} t^{(jl)} \boldsymbol{A}_T^{(l)}, \tag{27}$$

and the second one is the continuity of the stress vector $\boldsymbol{T}$ with components $T_j = \sigma_{jl} n_l$. According to (11) this vector for a displacement $\boldsymbol{u}(\boldsymbol{r}, t)$ is equal to

$$\boldsymbol{T}\Big(\boldsymbol{u}(\boldsymbol{r}, t)\Big) = \lambda \boldsymbol{n}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \mu[\boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{n}) + (\boldsymbol{n} \cdot \boldsymbol{\nabla})\boldsymbol{u}]. \tag{28}$$

Continuity of the vector $\boldsymbol{T}$ is equivalent to the equation

$$\boldsymbol{B}^{(j)} + \sum_{l=1}^{3} r^{(jl)} \boldsymbol{B}_R^{(l)} = \sum_{l=1}^{3} t^{(jl)} \boldsymbol{B}_T^{(l)}, \tag{29}$$

where the vector $\boldsymbol{B}$ is defined as

$$\boldsymbol{B} = -i \exp(-i\boldsymbol{k}\boldsymbol{r}) \boldsymbol{T}\Big(\boldsymbol{A} \exp(i\boldsymbol{k}\boldsymbol{r})\Big) = \lambda \boldsymbol{n}(\boldsymbol{k} \cdot \boldsymbol{A}) + \mu[\boldsymbol{k}(\boldsymbol{A} \cdot \boldsymbol{n}) + \boldsymbol{A}(\boldsymbol{n} \cdot \boldsymbol{k})], \tag{30}$$

for every plane wave $\boldsymbol{A} \exp(i\boldsymbol{k}\boldsymbol{r})$.

Note that the condition (29) makes it possible to continue the wave equation (13) from $z < 0$ to $z > 0$. If it is not satisfied, the differentiation of σ_{ij} in (13) creates $\delta(z)$-function and the wave equation becomes inhomogeneous [9] with a source term at the interface $z = 0$.

To find reflection and transmission amplitudes we need to multiply both equations (27) and (29) by three mutually orthogonal unit vectors to get in general 6 equations for 6

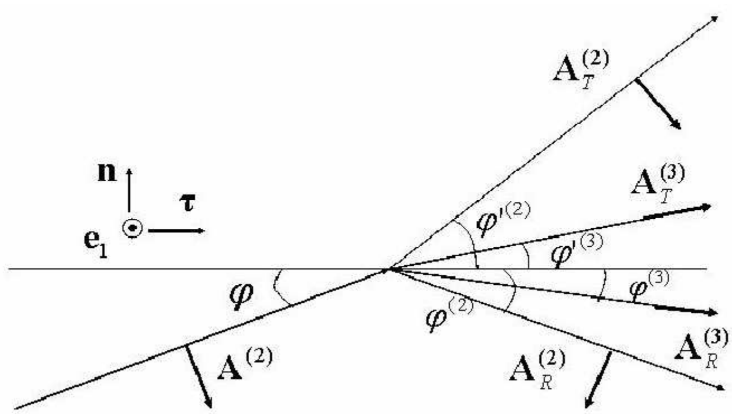

FIG. 1: Reflection of a transverse wave $\boldsymbol{A}^{(2)}$ from an interface between two different isotropic media. Reflected and refracted waves contain two modes: $\boldsymbol{A}_R^{(2,3)}$ and $\boldsymbol{A}_T^{(2,3)}$ respectively. The $\boldsymbol{A}_R^{(2)}$ mode goes at specular grazing angle $\varphi^{(2)} = \varphi$, the longitudinal mode $\boldsymbol{A}_R^{(3)}$ goes at grazing angle $\varphi^{(3)} < \varphi^{(2)}$.

unknowns. It is convenient to choose the right triple the vectors $\boldsymbol{\tau}$, $\boldsymbol{e}^{(1)}$, $\boldsymbol{n}$, as shown in Fig, 1, where the vector $\boldsymbol{e}^{(1)}$ is perpendicular to the incidence plane and in Fig. 1 points toward the reader.

Before calculations of the amplitudes of all the reflected and refracted waves we can easily understand what are all the angles. The grazing angle φ of the wave with the wave vector $\boldsymbol{k}$ is defined via relation $\cos\varphi = \boldsymbol{\tau}\cdot\boldsymbol{k}/k = k_\parallel/k = k_\parallel V/\omega$. Since $k_\parallel$ and ω are identical for all the waves therefore the value $\cos\varphi^{(j)}/V^{(j)}$ are also the same for all the waves. And because of (22) we can write

$$\frac{\cos\varphi}{V^{(j)}} = \frac{\cos\varphi^{(1,2)}}{c_t} = \frac{\cos\varphi^{(3)}}{c_l} = \frac{\cos\varphi'^{(1,2)}}{c'_t} = \frac{\cos\varphi'^{(3)}}{c'_l}, \tag{31}$$

where $\varphi^{(i)}$ $\varphi'^{(i)}$ denote grazing angle for respectively reflected and transmitted waves of mode i, and φ without indices denotes the grazing angle of the incident wave.

To find directions of polarization after reflection is very easy and we leave it as an exercise for the reader to check that with account of (23)

$$\boldsymbol{A}_R^{(2)} = [\boldsymbol{\kappa}_R^{(2)}\times\boldsymbol{e}^{(1)}] = -\frac{k_\perp^{(2)}\boldsymbol{\tau} + k_\parallel\boldsymbol{n}}{k^{(2)}}, \quad \boldsymbol{A}_R^{(3)} = \frac{-k_\perp^{(3)}\boldsymbol{n} + k_\parallel\boldsymbol{\tau}}{k^{(3)}}. \tag{32}$$

1. Reflection of $\boldsymbol{A}^{(1)}$ mode

The simplest is reflection and refraction of $\boldsymbol{A}^{(1)}$ mode. Its polarization is $\boldsymbol{e}^{(1)}$. After multiplication of equations (27) and (29) by $\boldsymbol{e}^{(1)}$ we get

$$1 + r^{(11)} = t^{(11)}, \qquad \mu_1(1 - r^{(11)})k_\perp = \mu_2 t^{(11)}k'_\perp, \tag{33}$$

from which it immediately follows that

$$r^{(11)} = \frac{\mu k_\perp - \mu' k'_\perp}{\mu k_\perp + \mu' k'_\perp}, \tag{34}$$

where $k_\perp = \sqrt{\omega^2/c_t^2 - k_\parallel^2}$, and $k'_\perp = \sqrt{\omega^2/c_t'^2 - k_\parallel^2}$. We see that this mode is reflected specularly and no other modes are created.

2. Reflection of $\boldsymbol{A}^{(2)}$ mode, and the mode conversion.

The more interesting is the case of the incident $\boldsymbol{A}^{(2)}$ mode shown in Fig. 1. Its reflection and refraction creates longitudinal mode $\boldsymbol{A}_R^{(3)}$, and because the speed c_l of $\boldsymbol{A}_R^{(3)}$ is larger than the speed c_t of the specularly reflected $\boldsymbol{A}_R^{(2)}$ mode, the grazing angle $\varphi^{(3)}$ is

less than $\varphi^{(2)}$. Therefore we can expect that at some angle $\varphi = \varphi_c$ of the incident wave, the angle $\varphi^{(3)}$ becomes zero, which means that the longitudinal mode ceases to propagate in the direction $z < 0$. Since according to (31) $\cos\varphi^{(3)} = (c_l/c_t)\cos\varphi \le 1$, we find that $\varphi_c = \arccos(c_t/c_l)$. The similar considerations are applicable to the refracted waves, and we can expect that at $\varphi < \arccos(\max(c_t/c_l, c'_t/c'_l)$ there appears a purely longitudinal surface wave propagating along the interface.

To find amplitudes of the reflected and refracted modes we have to multiply the two equations, (27) and (29), by $\boldsymbol{n}$ and $\boldsymbol{\tau}$. As a result we get a linear system of four equations for 4 unknown $r^{(22)}$, $r^{(23)}$, $t^{(22)}$ and $t^{(23)}$, which can be solved analytically. However it is a boring job, so it is better to pass it to computer.

The analytical solution can be found for reflection from a free surface, where we have a single boundary condition

$$\boldsymbol{B}^{(2)} + r^{(22)}\boldsymbol{B}_R^{(2)} + r^{(23)}\boldsymbol{B}_R^{(3)} = 0. \tag{35}$$

In this case we have only two reflected waves and multiplication of (35) by $\boldsymbol{n}$ and $\boldsymbol{\tau}$ with account of (30) and (32) gives only two equations

$$-\frac{2k_\perp^{(2)}k_\|}{k^{(2)}}(1 - r^{(22)}) + r^{(23)}\frac{k^{(2)2} - k_\|^2}{k^{(3)}} = 0, \tag{36}$$

$$\frac{k^{(2)2} - 2k_\|^2}{k^{(2)}}(1 + r^{(22)}) - 2r^{(23)}\frac{k_\perp^{(3)}k_\|}{k^{(3)}} = 0. \tag{37}$$

Their solution is

$$r^{(23)} = \frac{k^{(3)}}{k^{(2)}}\frac{4k_\perp^{(2)}k_\|(k^{(2)2} - 2k_\|^2)}{4k_\perp^{(3)}k_\perp^{(2)}k_\|^2 + (k^{(2)2} - 2k_\|^2)^2}, \qquad r^{(22)} = \frac{4k_\perp^{(3)}k_\perp^{(2)}k_\|^2 - (k^{(2)2} - 2k_\|^2)^2}{4k_\perp^{(3)}k_\perp^{(2)}k_\|^2 + (k^{(2)2} - 2k_\|^2)^2}. \tag{38}$$

At $\varphi = \varphi_c$ the vector $\boldsymbol{\kappa}^{(3)}$ of the longitudinal wave propagation direction coincides with $\boldsymbol{\tau}$, and therefore the length $k^{(3)} = \omega/c_l$ of the wave vector $\boldsymbol{k}^{(3)}$ becomes equal to $k_\|$. When the grazing angle φ decreases below φ_c the value of $k_\|$ increases, but ω/c_l does not change. Therefore at $\varphi < \varphi_c$ we get $k_\| > \omega/c_l$, and $k_\perp^{(3)} = \sqrt{\omega^2/c_l^2 - k_\|^2}$ becomes imaginary. We can denote it $-iK_l$. With such a normal component of the wave vector the longitudinal wave, propagating along the free surface, becomes localized in the layer of thickness $l = 1/K_l$, where $K_l = \sqrt{k_\|^2 - \omega^2/c_l^2}$. In other words, it becomes longitudinal surface wave $\boldsymbol{u}_S^{(3)}$ (lower index S means surface) with complex polarization vector $\boldsymbol{A}_S^{(3)}$. But what a strange wave it is! Since the incident wave can have arbitrary ω and $k_\| < k = \omega/c_t$, the longitudinal surface wave with the same ω and $k_\|$, has the speed along the surface, (denote it $V_S^{(3)}$)

equal to $V_S^{(3)} = \omega/k_\| > c_t$, which means, that it is not the Rayleigh surface wave, because the speed c_R of the Rayleigh wave, as is well known, is less than c_t!

However, really, it is not strange. This longitudinal surface wave satisfies the same wave equation $\Omega^2 = E$ of (20), and has longitudinal polarization

$$\boldsymbol{A}_S^{(3)} = (k_\|\boldsymbol{\tau} - iK_l\boldsymbol{n})/\sqrt{k_\|^2 + K_l^2}, \tag{39}$$

i.e. its normal component is imaginary. It is not dangerous that this polarization is a complex vector. The displacement must have a real value, therefore the displacement with a complex polarization vector is

$$\boldsymbol{u}_S^{(3)} \propto \mathrm{Re}\left[(k_\|\boldsymbol{\tau} - iK_l\boldsymbol{n})e^{i\boldsymbol{k}_\|\boldsymbol{r}_\| + K_l z - i\omega t}\right] = \left[(k_\|\boldsymbol{\tau}\cos(\boldsymbol{k}_\|\boldsymbol{r}_\| - \omega t) + K_l\boldsymbol{n}\sin(\boldsymbol{k}_\|\boldsymbol{r}_\| - \omega t)\right]e^{K_l z}, \tag{40}$$

i.e. the phase of oscillations along vector $\boldsymbol{n}$ is shifted by $\pi/2$ with respect to oscillations along vector $\boldsymbol{\tau}$. The speed of the longitudinal wave along the interface, $V_S^{(3)} = \omega/k_\|$, can be arbitrary, though because of $k_\|^2 > \omega^2/c_l^2$, this speed lies in the interval $c_t < V_S^{(3)} < c_l$.

In a similar way we can define the transverse surface wave. It satisfies the wave equation $\Omega^2 = 0$ of (20), and has transverse complex polarization

$$\boldsymbol{A}_S^{(2)} = (k_\|\boldsymbol{n} + iK_t\boldsymbol{\tau})/\sqrt{k_\|^2 + K_t^2}, \tag{41}$$

where $K_t = \sqrt{k_\|^2 - \omega^2/c_t^2}$. The real displacement vector in it is

$$\boldsymbol{u}_S^{(2)} \propto \mathrm{Re}\left[(k_\|\boldsymbol{n} + iK_t\boldsymbol{\tau})e^{i\boldsymbol{k}_\|\boldsymbol{r}_\| + K_t z - i\omega t}\right] = \left[(k_\|\boldsymbol{n}\cos(\boldsymbol{k}_\|\boldsymbol{r}_\| - \omega t) - K_t\boldsymbol{\tau}\sin(\boldsymbol{k}_\|\boldsymbol{r}_\| - \omega t)\right]e^{K_t z}, \tag{42}$$

and the speed $V_S^{(2)}$ along the interface can be arbitrary but less than c_t.

When longitudinal wave is of the surface type, the reflection amplitude of the $\boldsymbol{A}_R^{(2)}$ mode according to (38) becomes

$$r^{(22)} = -\frac{(k^{(2)2} - 2k_\|^2)^2 - 4iK_l k_\perp^{(2)} k_\|^2}{(k^{(2)2} - 2k_\|^2)^2 + 4iK_l k_\perp^{(2)} k_\|^2}. \tag{43}$$

It is a unit complex number, therefore it describes the total reflection of the incident wave.

3. *Energy flux distribution between two reflected waves*

Because of energy conservation the energy flux density of the incident wave along the normal to the interface must be equal to the sum of energy flux densities of the reflected waves. Let's check, whether they are really equal.

When the displacement and therefore the stress tensor are real functions, the energy flux density of an elastic wave is described by a vector $\boldsymbol{j}$ with components

$$j_i = -\langle \sigma_{il} du_l/dt \rangle, \tag{44}$$

where $\langle F \rangle$ means averaging of the function F over time. We use displacement in the form of complex plane waves, but the energy flux density should have only a real value, therefore the Eq. (44) can be represented as [10]

$$j_i = -\frac{1}{2}\left[\sigma_{il}^* \frac{du_l}{dt} + \sigma_{il}\frac{du_l^*}{dt}\right] = -\mathrm{Re}\left[\sigma_{il}^*\frac{du_l}{dt}\right], \tag{45}$$

where $\mathrm{Re}(F)$ means real part of F, and $*$ means complex conjugation. We are interested in the flux density along the normal $\boldsymbol{n}$ to the interface, therefore we need to calculate $\boldsymbol{nj} = \omega\mathrm{Re}(i\boldsymbol{T}^*\boldsymbol{u})$, where we used (28). Taking into account the definition (30) we can represent the energy flux in the form

$$\frac{(\boldsymbol{j}\cdot\boldsymbol{n})}{\mu\omega} = \frac{1}{\mu}\mathrm{Re}(\boldsymbol{B}^*\cdot\boldsymbol{A}) = E\mathrm{Re}((\boldsymbol{A}^*\cdot\boldsymbol{n})(\boldsymbol{k}\cdot\boldsymbol{A}) + k_\perp). \tag{46}$$

In the case of the incident $\boldsymbol{A}^{(2)}$ mode its incident flux is $j^{(2)} = \mu\omega k_\perp^{(2)} = \rho\omega c_t^2 k_\perp^{(2)}$. Reflected fluxes of the two modes are $j_R^{(2)} = \rho c_t^2 |r^{(22)}|^2 k_\perp^{(2)}$, $j_R^{(3)} = \rho c_l^2 |r^{(23)}|^2 k_\perp^{(3)}$ Energy conservation law requires

$$|r^{(22)}|^2 + |r^{(23)}|^2 \frac{c_l^2 k_\perp^{(3)}}{c_t^2 k_\perp^{(2)}} = 1. \tag{47}$$

Substitution of (38) shows that this equation is satisfied.

When the longitudinal wave becomes of surface type, it does not produce a flux from the surface. Therefore the law of energy conservation (47) reduces to

$$|r^{(22)}|^2 = 1, \tag{48}$$

which, according to (43), is also satisfied.

B. Energy density of the longitudinal surface wave

There is also one interesting question: what the energy density is accumulated in the longitudinal surface wave. This question is interesting, because it is this wave can be important for predictions and estimation of magnitudes of the earthquakes.

The vector of energy density flux of a wave of mode $\boldsymbol{A}$ according to (45) can be represented as

$$\boldsymbol{J} = \mathrm{Re}(\rho\omega c_t^2 u_0^2 [E\boldsymbol{A}^*(\boldsymbol{k}^*\cdot\boldsymbol{A}) + \boldsymbol{k}^*]), \tag{49}$$

where we introduced an amplitude u_0 of the wave. Expression (49) is valid for real and complex wave vectors and polarizations. The absolute value of this flux for the incident transverse wave is

$$J = \rho\omega c_t^2 u_0^2 k. \tag{50}$$

For the surface longitudinal wave with account of its factor r^{23} we get

$$J_S^3(z) = \rho\omega u_0^2 |r^{23}|^2 c_l^2 k_\parallel \exp(2K_l z). \tag{51}$$

We do not know u_0, so we can find only ratio $Q = J_S^3(z=0)/J$. Substitution of r^{23} from

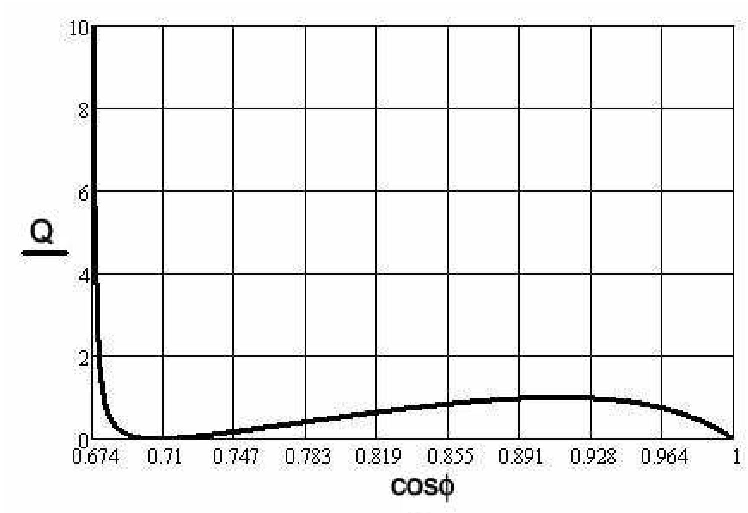

FIG. 2: Dependence of Q on $\cos\varphi$ calculated for $E = 1.2$. The left point on the abscise axis corresponds to critical $\cos\varphi_c = 0.674$, where $Q = 323.16$.

(38) gives

$$Q = \frac{k_\parallel}{k^{(2)}}\frac{c_l^2}{c_t^2}|r^{23}|^2 = \frac{k_\parallel}{k^{(2)}}\frac{16k_\perp^{(2)2}k_\parallel^2(k^{(2)2}-2k_\parallel^2)^2}{16K_l^2 k_\perp^{(2)2}k_\parallel^4 + (k^{(2)2}-2k_\parallel^2)^4}, \tag{52}$$

or

$$Q = \frac{\cos\varphi\sin^2(4\varphi)}{4(\cos^2\varphi - \cos^2\varphi_c)\sin^2(2\varphi) + \cos^4(2\varphi)}. \tag{53}$$

Dependence of this function on $\cos\varphi$ is shown in Fig. 2. We see that the highest energy density is accumulated in longitudinal surface wave, when φ is slightly less than φ_c. There is also a maximum at small angles where the ratio Q is close to unity.

C. The Surface Rayleigh wave

We considered above the two surface waves, which satisfy the wave equations, but cannot exist independently, because without the incident and reflected waves they do not

satisfy the boundary condition. The Rayleigh surface wave exists without the incident one, and its speed $c_R = \omega/k_\parallel < c_t$ is fixed. To get equation which determines this speed c_R we represent the boundary condition (35) in the form

$$\frac{1}{r^{(22)}}\boldsymbol{B}^{(2)} + \boldsymbol{B}_R^{(2)} + \frac{r^{(23)}}{r^{(22)}}\boldsymbol{B}_R^{(3)} = 0, \tag{54}$$

where $r^{(22)}$ and $r^{(23)}$ are given by (38). With (54) we can immediately find the speed of the Rayleigh surface wave. It corresponds to such $\omega/k_\parallel$, for which the first term in (54) is zero. Since $\boldsymbol{B}^{(2)} \neq 0$, therefore the first term is zero only when

$$\frac{1}{r^{(22)}} = 0. \tag{55}$$

In such a case the incident wave disappears, and the whole wave field contains only two waves propagating along the free surface.

Let's remind that a similar trick helps to find bound states of particles in quantum mechanics. Reflection of a particle from a one dimensional potential well is described in asymptotic region $x \to -\infty$ by the wave function $\exp(ikx)+r(k)\exp(-ikx)$, where $r(k)$ is a reflection amplitude. This wave function can be also represented as $(1/r(k))\exp(ikx)+\exp(-ikx)$. In bound states the wave function at $x \to -\infty$ has asymptotics $\exp(-Kx)$, where $-K^2$ is proportional to the bound level E_b. To find K we need to solve equation $1/r(k) = 0$, which annuls the incident wave $\exp(ikx)$. Every root of this equation $k_n = -iK_n$ corresponds to n-th bound level $E_{bn} \propto -K_n^2$. In that respect the Rayleigh surface wave is a bound state of elastic waves,

After this digression we go back. From (43) it follows that (55) is satisfied, if

$$4k_\perp^{(3)}k_\perp^{(2)}k_\parallel^2 + (k^{(2)2} - 2k_\parallel^2)^2 = 0. \tag{56}$$

It is important to note that the third term in (54) does not disappear though it also contains the factor $1/r^{(22)}$. It does not disappear because $r^{(23)}$ and $r^{(22)}$ according to (38) have the similar denominators, and they cancel each other in the ratio $r^{(23)}/r^{(22)}$. In fact the amplitudes $r^{(23)}$ and $r^{(22)}$ play equal roles, so instead of (54) we can write

$$\frac{1}{r^{(23)}}\boldsymbol{B}^{(2)} + \frac{r^{(22)}}{r^{(23)}}\boldsymbol{B}_R^{(2)} + \boldsymbol{B}_R^{(3)} = 0, \tag{57}$$

and seek solution of the equation $1/r^{(23)} = 0$. The result will be the same.

Let's denote the speed of the wave propagation, $\omega/k_\parallel$, along the interface by c_R (speed of the Rayleigh wave), and its ratio to c_t by $x = c_R/c_t$. Since $k_\perp^{(2)}$ and $k_\perp^{(3)}$ in surface

waves are to be imaginary then $k_\perp^{(2)2} = \omega^2/c_t^2 - k_\parallel^2 < 0$, $k_\perp^{(3)2} = \omega^2/c_l^2 - k_\parallel^2 < 0$, and the equation (56) is reduced to

$$4\sqrt{1-x^2}\sqrt{1-\varsigma^2 x^2} = (2-x^2)^2, \tag{58}$$

where $\varsigma = c_t/c_l$. This equation and its solution are well known and can be found in all the textbooks on elastic waves. It can be solved even analytically for arbitrary ς, because it is equivalent to an algebraic equation of 4-th order with respect to variable $z = 1 - x^2$.

When (58) is satisfied, the total displacement vector of the Rayleigh wave becomes

$$\boldsymbol{u}_R \propto \boldsymbol{u}_S^{(2)} + \frac{r^{(23)}}{r^{(22)}}\boldsymbol{u}_S^{(3)}, \tag{59}$$

Substitution of (38), (40) and (42) into (59) with the same $\omega/k_\parallel = c_R$ gives [11]

$$\boldsymbol{u}_R \propto \boldsymbol{n}\cos(\boldsymbol{k}_\parallel \boldsymbol{r}_\parallel - \omega t)[2q_t q_l e^{K_l z} - (1+q_t^2)e^{K_t z}] - \boldsymbol{\tau}\sin(\boldsymbol{k}_\parallel \boldsymbol{r}_\parallel - \omega t)q_t[2e^{K_l z} - (1+q_t^2)e^{K_t z}]. \tag{60}$$

where $K_{t,l} = k_\parallel\sqrt{1 - c_R^2/c_{t,l}^2}$, and $q_{t,l} = K_{t,l}/k_\parallel = \sqrt{1 - c_R^2/c_{t,l}^2}$.

In a similar way we can find the Stoneley surface wave propagating along the interface between two isotropic media. Though there are no principal difficulties, we do not consider it here because of technical complications.

From (30) we can immediately conclude that the surface waves with polarization along the surface and perpendicular to direction of propagation do not exist, because the continuity of the stress vector $\boldsymbol{B}$ requires continuity of the normal derivative of the displacement vector, which cannot be satisfied.

IV. WAVES IN ANISOTROPIC MEDIA ($\xi \neq 0$)

In isotropic media it was natural to describe polarizations in an orthogonal basis $\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)}$ and $\boldsymbol{\kappa} = \boldsymbol{k}/k$, which constitutes the right hand triple of unit vectors. The choice of $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$ had some freedom because these two vectors can be rotated by an arbitrary angle around $\boldsymbol{\kappa}$. It was only in studying of reflection from an interface, where orientation of $\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)}$ was fixed by the plane of incidence. In anisotropic media besides $\boldsymbol{\kappa}$ we have also vector $\boldsymbol{a}$, so for orientation of vectors $\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)}$ it is better to take the plane of vectors $\boldsymbol{\kappa}$ and $\boldsymbol{a}$ into account, choosing $\boldsymbol{e}^{(2)}$ in the plane, and $\boldsymbol{e}^{(1)} = [\boldsymbol{e}^{(2)} \times \boldsymbol{\kappa}]$ perpendicular to it.

After substitution of (19) into (15) and multiplication by three vectors $\boldsymbol{e}^{(1)}$, $\boldsymbol{e}^{(2)}$ and $\boldsymbol{\kappa} = \boldsymbol{k}/k$ we obtain the system of linear equations for $\alpha^{(1,2)}$, and β. It has a solution, if its

determinant is equal to zero. This condition gives three possible possible speeds $V^{(i)}(k)$ (i=1,2,3) for the three wave modes.

The simplest equation is obtained after multiplication of the (15) by $\boldsymbol{e}^{(1)}$. The result is

$$\Omega^2\alpha^{(1)} + \xi(\boldsymbol{\kappa}\cdot\boldsymbol{a})^2\alpha^{(1)} = 0. \tag{61}$$

This Eq. is equivalent to

$$\Omega^2 + \xi(\boldsymbol{\kappa}\cdot\boldsymbol{a})^2 = 0, \tag{62}$$

and it gives the speed of this transverse mode

$$V^{(1)} \equiv \omega/k = c_t\sqrt{1-\xi(\boldsymbol{\kappa}\cdot\boldsymbol{a})^2} = c_t\sqrt{1-\xi\cos^2\theta}, \tag{63}$$

where θ is the angle between vectors $\boldsymbol{\kappa}$ and $\boldsymbol{a}$. We see that this speed is less than c_t, and it changes with θ.

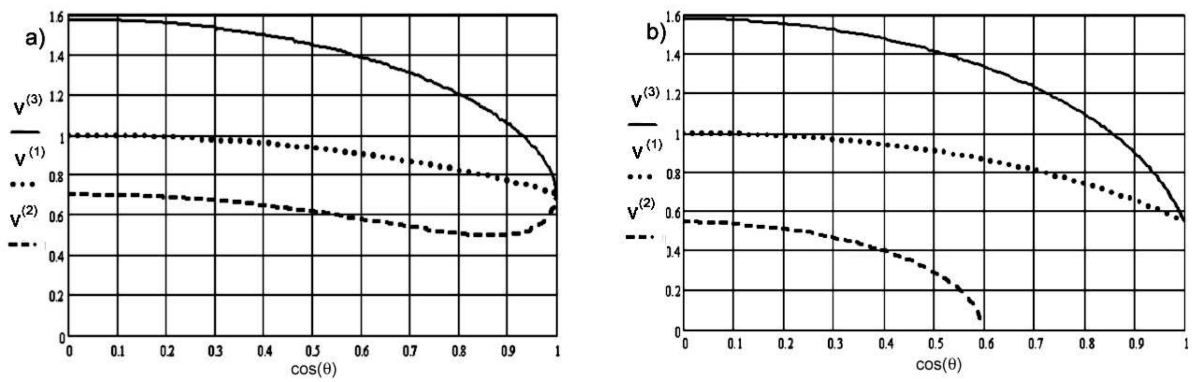

FIG. 3: The transverse, $V^{(1)}$, quasi transverse, $V^{(2)}$, and quasi longitudinal, $V^{(3)}$, speeds of elastic waves in an anisotropic media in dependence on $\cos\theta$, where θ is the angle between wave vector $\boldsymbol{k}$ and anisotropy vector $\boldsymbol{a}$, for $E = 1.5$ and two different anisotropy parameters a) $\xi = 0.5$; b) $\xi = 0.7$. The unity on ordinate axis corresponds to $c_t = \sqrt{\mu/\rho}$.

After multiplication of (15) by $\boldsymbol{e}^{(2)}$ and $\boldsymbol{\kappa}$ we obtain a system of two equations

$$(\Omega^2 + \xi)\alpha^{(2)} + 2\xi(\boldsymbol{a}\cdot\boldsymbol{e}^{(2)})(\boldsymbol{\kappa}\cdot\boldsymbol{a})\beta = 0, \tag{64}$$

$$[\Omega^2 - E + 4\xi(\boldsymbol{\kappa}\cdot\boldsymbol{a})^2]\beta + 2\xi(\boldsymbol{a}\cdot\boldsymbol{e}^{(2)})(\boldsymbol{\kappa}\cdot\boldsymbol{a})\alpha^{(2)} = 0, \tag{65}$$

where in (64) we used relation $(\boldsymbol{a}\cdot\boldsymbol{e}^{(2)})^2 + (\boldsymbol{\kappa}\cdot\boldsymbol{a})^2 = 1$. We see that polarizations along $\boldsymbol{e}^{(2)}$ and $\boldsymbol{\kappa}$ are not independent. They combine and create two new hybridized polarizations, which we call quasi transverse and quasi longitudinal modes and denote by $\boldsymbol{A}^{(2,3)}$ like in isotropic case.

The system (64,65) has solutions, if

$$[\Omega^2+\xi][\Omega^2-E+4\xi(\boldsymbol{\kappa}\cdot\boldsymbol{a})^2]-4\xi^2(\boldsymbol{\kappa}\cdot\boldsymbol{a})^2(\boldsymbol{a}\cdot\boldsymbol{e}^{(2)})^2=0, \tag{66}$$

From which it follows

$$2(\Omega^{(2,3)2}+\xi)=E+\xi[1-4(\boldsymbol{\kappa}\cdot\boldsymbol{a})^2]\mp\sqrt{\{E+\xi[1-4(\boldsymbol{\kappa}\cdot\boldsymbol{a})^2]\}^2+16\xi^2(\boldsymbol{\kappa}\cdot\boldsymbol{a})^2(\boldsymbol{a}\cdot\boldsymbol{e}^{(2)})^2}. \tag{67}$$

Since $(\boldsymbol{\kappa}\cdot\boldsymbol{a})^2=\cos^2\theta$, and $(\boldsymbol{a}\cdot\boldsymbol{e}^{(2)})^2=\sin^2\theta$ then, because $\Omega^{(2,3)2}=V^{(2,3)2}/c_t^2-1$, we get that the equation (67) is equivalent to

$$V^{(2,3)}=c_t\sqrt{1-\xi+\frac{E+\xi(1-4\cos^2\theta)\mp\sqrt{[E+\xi(1-4\cos^2\theta)]^2+4\xi^2\sin^2(2\theta)}}{2}}. \tag{68}$$

At $\xi\to 0$ their values are

$$V^{(2)}\approx c_t\left(1-\frac{\xi}{2}\right)-O(\xi^2),\qquad V^{(3)}\approx c_l-2\frac{\xi}{c_l}\cos^2\theta+O(\xi^2), \tag{69}$$

where $O(\xi^2)$ denotes a small number proportional to ξ^2. So $V^{(2)}$ can be called quasi transverse and $V^{(3)}$ — quasi longitudinal speed.

All the speeds $V^{(1)}$, $V^{(2)}$ and $V^{(3)}$ depend on angle θ. This dependence is shown in Fig. 3. We see that if the anisotropy parameter ξ is sufficiently large some modes at small angles θ cease to propagate, because their speed, as is shown in Fig. 3b) for quasi transverse mode, does not exist. This speed becomes imaginary, therefore the wave number of the mode, $k^{(2)}=\omega/V^{(2)}$, also becomes imaginary, and the wave does not propagate. Of course it corresponds to too large anisotropy parameter. Since $E=1.5$ then $\xi=0.7$ means that anisotropy energy ζ is larger than the Lamé index λ, and in some directions the higher is deformation the less is the elastic energy, which is nonphysical. For smaller ξ the speed V_2 is at no angle imaginary.

From (64) and (65) it follows that polarization of propagating quasi transverse, $\boldsymbol{A}^{(2)}$, and quasi longitudinal, $\boldsymbol{A}^{(3)}$, modes are

$$\boldsymbol{A}^{(2)}=\frac{\xi\sin(2\theta)\boldsymbol{e}^{(2)}-(\Omega^{(2)2}+\xi)\boldsymbol{\kappa}}{\sqrt{(\Omega^{(2)2}+\xi)^2+\xi^2\sin^2(2\theta)}},\qquad \boldsymbol{A}^{(3)}=\frac{(\Omega^{(3)2}+\xi)\boldsymbol{\kappa}-\xi\sin(2\theta)\boldsymbol{e}^{(2)}}{\sqrt{(\Omega^{(3)2}+\xi)^2+\xi^2\sin^2(2\theta)}}. \tag{70}$$

At small ξ they, as can be expected, are:

$$\boldsymbol{A}^{(2)}\approx\boldsymbol{e}^{(2)}+\frac{\xi}{E}\sin(2\theta)\boldsymbol{\kappa},\qquad \boldsymbol{A}^{(3)}\approx\boldsymbol{\kappa}-\frac{\xi}{E}\sin(2\theta)\boldsymbol{e}^{(2)}. \tag{71}$$

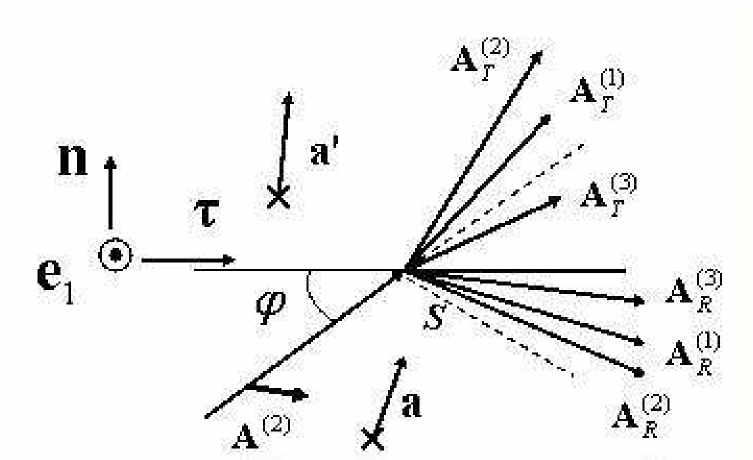

FIG. 4: Splitting of reflected and refracted waves at an interface between two different anisotropic media when the incident ia a wave of quasi transverse mode $\boldsymbol{A}^{(2)}$. S denotes the specular direction. Anisotropy vector $\boldsymbol{a}$ has such a direction that the speed of reflected $\boldsymbol{A}_R^{(2)}$ mode is higher than that of the incident one.

A. Reflection from an interface

Reflection of waves from an interface in anisotropic media is in general characterized by trirefringency, as was correctly pointed out in [1]. An incident wave at an interface in general splits into three reflected and three refracted waves, and no wave is reflected specularly. In Fig. 4 we present the scheme of reflection and refraction of a quasi transverse wave from an interface between two anisotropic media with different anisotropy vectors $\boldsymbol{a}$ and $\boldsymbol{a}'$ and different parameter ρ, λ, μ and ζ. The anisotropy vectors in general are not in the incidence plane. In Fig. 4 they are inclined down, so the reader sees their tails denoted by crosses.

The grazing angles $\varphi^{(i)}$, $\varphi'^{(i)}$ (angles between wave vectors $\boldsymbol{k}_{R,T}^{(i)}$ of reflected and refracted modes $\boldsymbol{A}_{R,T}^{(i)}$ and the unit vector $\boldsymbol{\tau}$) in the case when the incident wave is of mode j, are determined from the relations equivalent to (31):

$$\frac{\cos\varphi}{V^{(j)}}=\frac{\cos\varphi^{(i)}}{V_R^{(i)}}=\frac{\cos\varphi'^{(i)}}{V_T^{(i)}}. \qquad (72)$$

The value of the speed of a wave depends on the angle θ between the direction of its propagation and the anisotropy vector $\boldsymbol{a}$. It may happen that after reflection all the speeds are higher that the speed $V^{(j)}$ of the incident wave. Then the grazing angles of all the waves become less that that of the incident one as is shown in Fig. 5a), and we can expect that at some critical angle $\varphi=\varphi_c$ all the reflected and transmitted waves will

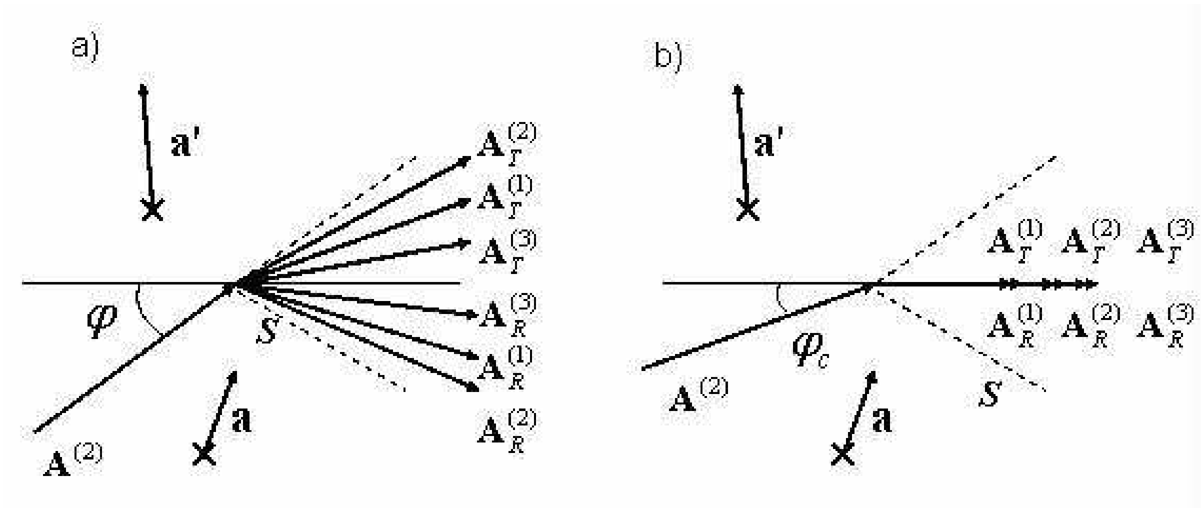

FIG. 5: Reflection, refraction and splitting of waves at an interface between two different anisotropic media, when the speeds of refracted waves are higher than that of the incident quasi transverse wave. a) The grazing angle of the incident wave is sufficiently high, and all the created waves are able to propagate in z direction. b) Unreal situation, when the incident quasi transverse wave is transformed into the surface wave containing all the three modes.

accumulate into a single surface wave as is shown in Fig. 5b).

Physically such a result is unacceptable, because the incident plane wave gives the energy flux toward the interface, therefore the energy must accumulate in the surface wave and the surface wave amplitude should increase with the time exponentially. We are dealing with stationary waves, therefore exponentially growing functions are excluded from our solutions.

We should look what is wrong in our logic, considering an example, in which everything can be solved analytically. The analytical solution can be found in the case of reflection of a quasi transverse wave from a free surface, when anisotropy vector lies in the incidence plane, as is shown in Fig. 6. In this case we have only two reflected modes: quasi transverse and quasi longitudinal ones, and to find their reflection amplitudes we need to solve only system of two linear equations.

B. Reflection of quasi transverse wave from a free surface, when anisotropy vector is in the incidence plane

Let's consider reflection of a plane wave of quasi transverse mode $\boldsymbol{A}^{(2)}$ from a free surface, when the anisotropy vector has such a direction, that the reflected quasi transverse

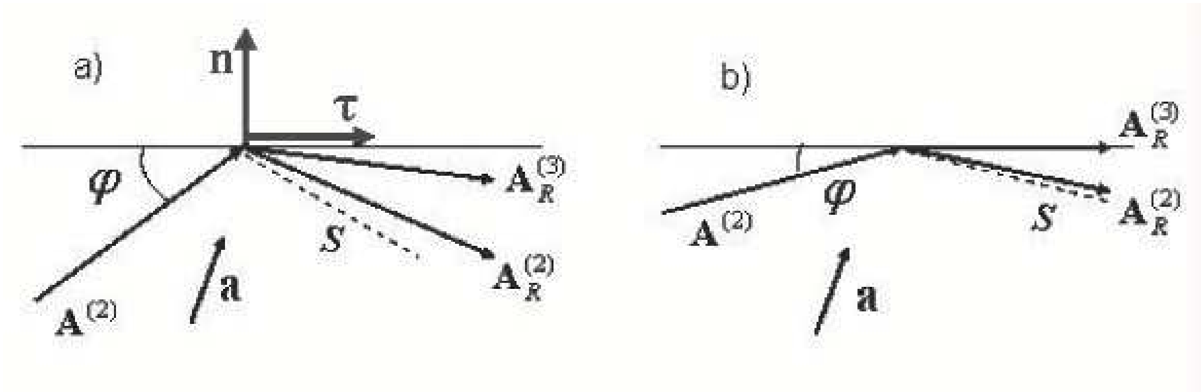

FIG. 6: Reflection of a quasi transverse wave from a free surface, when anisotropy vector lies in the incidence plane, and the reflected wave speed is higher than that of the incident one. Reflection is accompanied with creation of quasi longitudinal wave. a) The grazing angle of the incident wave is sufficiently large and both reflected waves can propagate in z direction. b) The grazing angle of the incident wave is sufficiently small, and quasi longitudinal wave propagates only along the surface.

wave has higher speed than the incident one.

The angles of reflected waves are determined by (31)

$$\frac{\cos\varphi}{V^{(2)}(\theta)}=\frac{\cos\varphi^{(2)}}{V_R^{(2)}(\theta^{(2)})}=\frac{\cos\varphi^{(3)}}{V_R^{(3)}(\theta^{(3)})}, \tag{73}$$

where θ and $\theta^{(2,3)}$ are the angles between $\boldsymbol{a}$ and directions of propagation $\boldsymbol{\kappa}$ of the incident and $\boldsymbol{\kappa}^{(2,3)}$ of the reflected waves respectively. In these equations we do not know $V_R^{(i)}(\theta^{(i)})$, therefore we cannot directly find $\varphi^{(i)}$. Instead we have to use these equations to find both $\varphi^{(i)}$ and $V_{ir}(\theta^{(i)})$ simultaneously.

Let's denote $\boldsymbol{a}=\boldsymbol{\tau}\cos\varphi_a+\boldsymbol{n}\sin\varphi_a$, $\boldsymbol{\kappa}=\boldsymbol{\tau}\cos\varphi+\boldsymbol{n}\sin\varphi$ and $\boldsymbol{\kappa}^{(i)}=\boldsymbol{\tau}\cos\varphi^{(i)}-\boldsymbol{n}\sin\varphi^{(i)}$ then $\cos\theta=\boldsymbol{a}\cdot\boldsymbol{\kappa}$ and $\cos\theta^{(i)}=\boldsymbol{a}\cdot\boldsymbol{\kappa}^{(i)}$. Substitution of $\cos\theta$ and $\cos\theta^{(i)}$ into (68) and after that into (73) gives transcendent equations that can be solved numerically.

The result of calculations for $\cos\varphi_a=0.4$, $\xi=0.4$ and $E=1.5$ are shown in Fig-s 7 and 8. Fig. 7 shows dependence of all the speeds on $\cos\varphi$, and Fig. 8 shows dependence of $\cos\varphi^{(i)}$ on $\cos\varphi$.

We see that at $\cos\varphi>0.92$ no reflected wave can propagate. What does happen there is the most interesting question!

At $\cos\varphi<0.5$, where both reflected waves do really exist, we can find their reflection amplitudes. For that we have to solve the boundary condition equation

$$\boldsymbol{B}^{(2)}+r^{(22)}\boldsymbol{B}_R^{(2)}+r^{(23)}\boldsymbol{B}_R^{(3)}=0, \tag{74}$$

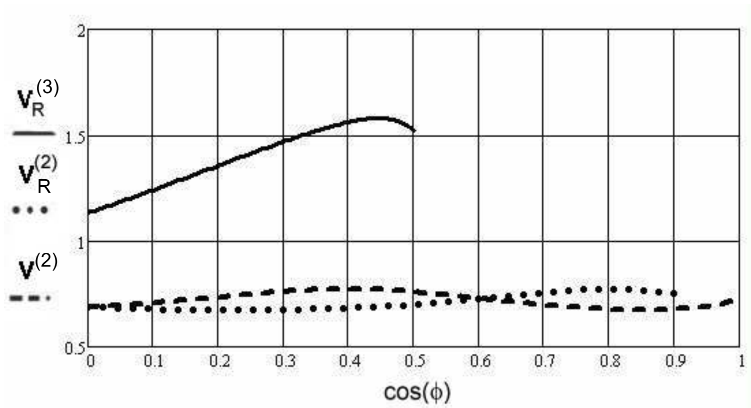

FIG. 7: Dependence of speeds of the quasi transverse incident, $V^{(2)}$, quasi transverse reflected, $V_R^{(2)}$ and quasi longitudinal reflected $V_R^{(3)}$ speeds on $\cos\varphi$ of the grazing incidence angle, when $(\boldsymbol{a}\cdot\boldsymbol{\tau}) = 0.4$; $\xi = 0.4$ and $E = 1.5$. We see that in some range of $\cos\varphi$ both reflected speeds are higher than that of the incident one.

where $\boldsymbol{B}$ is defined like in (30):

$$\boldsymbol{B} = -i\exp(-i\boldsymbol{kr})\boldsymbol{T}\Big(\boldsymbol{A}\exp(i\boldsymbol{kr})\Big) = (\boldsymbol{n}\cdot\boldsymbol{k})\boldsymbol{A} + \boldsymbol{k}(\boldsymbol{n}\cdot\boldsymbol{A}) + (E-1)\boldsymbol{n}(\boldsymbol{A}\cdot\boldsymbol{k}) -$$

$$-\xi\{\boldsymbol{a}[(\boldsymbol{n}\cdot\boldsymbol{k})(\boldsymbol{a}\cdot\boldsymbol{A}) + (\boldsymbol{a}\cdot\boldsymbol{k})(\boldsymbol{n}\cdot\boldsymbol{A})] + (\boldsymbol{n}\cdot\boldsymbol{a})[(\boldsymbol{a}\cdot\boldsymbol{k})\boldsymbol{A} + \boldsymbol{k}(\boldsymbol{a}\cdot\boldsymbol{A})]\}. \qquad (75)$$

Multiplying (74) by $\boldsymbol{n}$ and $\boldsymbol{\tau}$, we obtain 2 equations for two reflection amplitudes. Let's denote $\beta = (\boldsymbol{n}\cdot\boldsymbol{B}^{(2)})$, $\delta = (\boldsymbol{\tau}\cdot\boldsymbol{B}^{(2)})$, $\beta_i = (\boldsymbol{n}\cdot\boldsymbol{B}_R^{(i)})$ and $\delta_i = (\boldsymbol{\tau}\cdot\boldsymbol{B}_R^{(i)})$ where i=2,3, then Eq. (74) can be represented in matrix form

$$\begin{pmatrix}\beta\\ \delta\end{pmatrix} + \begin{pmatrix}\beta_2 & \beta_3\\ \delta_2 & \delta_3\end{pmatrix}\begin{pmatrix}r^{(22)}\\ r^{(23)}\end{pmatrix} = 0, \qquad (76)$$

and its solution is elementary. We do not represent the final analytical result because it does not look sufficiently informative.

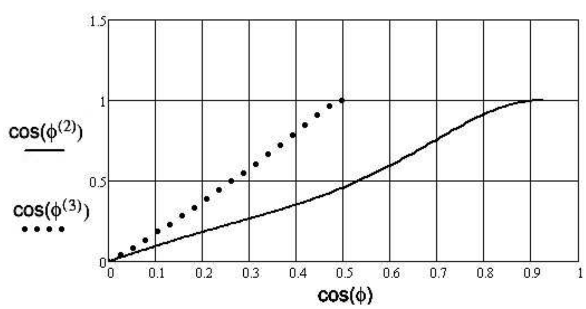

FIG. 8: Dependence of $\cos\varphi^{(2)}$ and $\cos\varphi^{(3)}$ on $\cos\varphi$. We see that both $\cos\varphi^{(2)}$ and $\cos\varphi^{(3)}$ reach unity at $\cos\varphi < 1$. It looks as if both waves become of surface type, when the incident one still remains to be the plain wave.

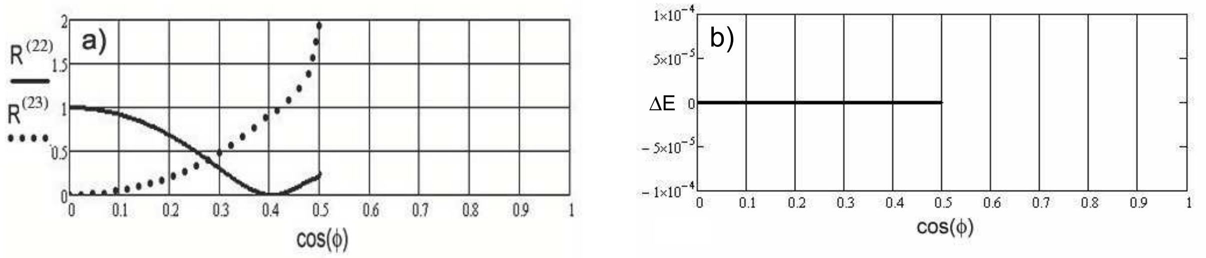

FIG. 9: Dependence of reflectivities $R^{(22)} = |r^{(22)}|^2$ of quasi transverse and $R^{(23)} = |r^{(23)}|^2$ of quasi longitudinal waves on $\cos\varphi$. b) The difference of energies ΔE of the incident and reflected waves. We see that all the values are well calculated only up to $\cos\varphi = 0.5$, where longitudinal wave becomes of the surface type.

The numerically calculated dependence of reflectivities $R^{(22)} = |r^{(22)}|^2$ and $R^{(23)} = |r^{(23)}|^2$ on $\cos\varphi$ is shown in the left panel of Fig. 9. The correctness of calculations is supported by the panel b), which demonstrates the energy conservation at reflection. All the calculations are possible only up to $cos\varphi = 0.5$. Above this value the the wave vector of the quasi longitudinal wave becomes complex, and equation (73) has no solutions.

C. Waves propagation near a free surface

To see what happens, above $cos\varphi = 0.5$ we have to make calculations differently. At the interface there are two conserved values: the frequency ω and the wave number $k_\parallel$ along the interface. It is worth to divide both parts of the Eq. (14) by $\mu k_\parallel^2$, to introduce the value

$$\Upsilon = \frac{\omega^2}{c_t^2 k_\parallel^2} = \frac{\omega^2}{k^2 c_t^2 \cos^2\varphi} = \frac{V^{(2)2}(\cos\theta)}{c_t^2\cos^2\varphi}, \tag{77}$$

and the dimensionless wave vector $\tilde{\boldsymbol{k}} = \boldsymbol{k}/k_\parallel = \boldsymbol{\tau} + q\boldsymbol{n}$, where $q = k_\perp/k_\parallel$. After that the Eq. (14) is transformed to

$$\left[\Upsilon - 1 - q^2 + \xi(\tilde{\boldsymbol{k}}\cdot\boldsymbol{a})^2\right]\boldsymbol{A} = E\tilde{\boldsymbol{k}}(\tilde{\boldsymbol{k}}\cdot\boldsymbol{A}) -$$

$$-\xi\left(\boldsymbol{a}[(1+q^2)(\boldsymbol{a}\cdot\boldsymbol{A}) + (\tilde{\boldsymbol{k}}\cdot\boldsymbol{a})(\tilde{\boldsymbol{k}}\cdot\boldsymbol{A})] + \tilde{\boldsymbol{k}}(\tilde{\boldsymbol{k}}\cdot\boldsymbol{a})(\boldsymbol{a}\cdot\boldsymbol{A})]\right). \tag{78}$$

This equation describes propagation of waves near any, even fictitious, surface, and it is valid also near the interface. With it we do not speak about incident and reflected waves. We look for all possible solutions near the interface and select those which correspond to our physics. Solution of Eq. (14), gave us wave modes and their speeds, solution of (78) will give wave modes and their q or $k_\perp$. We select in between them, say, one wave with

positive and two waves with negative q. They correspond to the incident and reflected waves. And we find such a superposition of these waves that satisfies the boundary condition (74). Thus we obtain the result absolutely equivalent to that obtained with Eq. (14). In the case of a real $k_\perp$ the value of q is $\mathrm{tg}\varphi$, but q can be also defined for arbitrary complex $k_\perp$, and this is the benefit of the Eq. (78)

All the waves, incident, reflected or surface ones, should satisfy this equation for the given value of Υ, which is determined by the grazing angle φ of the incident wave and by direction of the vector $\boldsymbol{a}$. Polarization vector $\boldsymbol{A}$ is represented as

$$\boldsymbol{A} = \alpha\boldsymbol{n} + \beta\boldsymbol{\tau}. \tag{79}$$

To find α and β we multiply both parts of Eq. (78) by $\boldsymbol{n}$ and $\boldsymbol{\tau}$, and obtain a system of two linear homogeneous equations, which has solution, when its determinant is equal to zero. The resulting equation is a polynomial of the 4-th order in powers of q, and it has 4 roots.

For instance, calculations for $\cos\varphi = 0.3$, which is below 0.5, give all the roots q to be real. Two of them are positive: $q_1 = 3.18$, $q_4 = 2.47$; and the other two are negative: $q_2 = -3.6$, $q_3 = -1.4$ (such numerations of the roots is for further convenience). These roots determine all the possible waves near the surface. The positive roots correspond to waves incident on the surface, and the negative ones correspond to waves going away from the surface. The root q_1 corresponds to the given $\cos\varphi = 0.3$ of the incident quasi transverse wave $\boldsymbol{A}^{(2)}$. The root q_4 shows that, if the incident wave were quasi longitudinal one, its grazing angle would be $\cos\varphi = 0.37$. The negative roots are related to the reflected waves: q_2 to the quasi transverse, $\boldsymbol{A}_R^{(2)}$, and q_3 — to the quasi longitudinal, $\boldsymbol{A}_R^{(3)}$, ones.

When $\cos\varphi = 0.6 > 0.5$ the two roots, q_3 and q_4, related to quasi longitudinal waves become complex conjugate: $q_{3,4} = 0.32 \mp 0.77i$. We have to take into consideration only q_3, which is related to quasi longitudinal surface wave. This root has correct sign of the imaginary part, which warrants an exponential decay of the wave away from the interface in the half space $z < 0$. However it does contain also a real part, which seems to make this root unacceptable. In particle physics the wave function $\psi \propto \exp(iq'z + q''z)$ at $z < 0$ means that there is a flux of particles $j \propto q' \exp(2q''z)$ toward the surface, which increases exponentially from $z = -\infty$, and shows that during propagation from $z = -\infty$ toward $z = 0$ the particles are created from nothing. Intuitively we expect the same of elastic waves. However elastic waves behave differently, and because of that

we call their properties counter intuitive. The numerical calculations of the energy flux according to (45) show that the surface quasi longitudinal wave does not create energy flux notwithstanding that its $k_\perp^{(3)}$ has a complex value.

The most terrible situation seems to occur after $\cos\varphi = 0.92$. (It is not a fundamental constant. It depends on direction of the anisotropy vector $\boldsymbol{a}$ and on values of parameters ξ and E). At some critical $\varphi_{c1} \approx 0.921352$ the value of q_2 and therefore of $k_\perp^{(2)}$ become zero, i.e. our anxieties came true! The incident plane wave turns into a surface one! However above this critical point the quasi transverse mode does not become of a surface type. Its q_2 and therefore $k_\perp^{(2)}$ do not acquire a negative imaginary part. Instead $k_\perp^{(2)}$ remains real but changes its sign!

Intuitively we can expect that after reverse of the sign of q_2 the wave becomes propagating toward the surface. Such a wave should carry the energy also toward the surface. Nothing like that! We found that the energy flux of this mode did not change its sign. Reflected energy flux related to this mode remains completely equal to the incident flux and opposite in direction. It can be understood because the energy flux depends not solely on the wave vector $\boldsymbol{k}$ but also on polarization (or oscillation) direction $\boldsymbol{A}$ and anisotropy vector $\boldsymbol{a}$. The direct calculations show that we have no reason to worry nor about energy conservation, nor about boundary conditions. They both are satisfied at $\cos\varphi > \cos\varphi_{c1}$.

However it is not the end of the story. When we decrease φ below φ_{c1}, the value of $q_1 = \text{ tg}\varphi$ decreases and the energy flux of the incident wave decreases too. This is natural. Reflected flux decreases in the same way, though $q_2 > 0$ steadily increases. But there is a second critical point φ_{c2}, where $q_1 = q_2$, and the energy flux density of the incident wave becomes zero! After this point the roles of the two roots q_1 and q_2 do exchange. The incident wave gives the flux away from the surface, and the reflected wave — toward it. Of course it means that the incident wave of the mode $\boldsymbol{A}^{(2)}$ does not exist below φ_{c2}! All that leads us to an interesting conclusion, but before going to it let's discuss the surface waves on a free surface in an anisotropic media.

D. Surface waves

The first question is: whether the surface waves do exist? From the very beginning it was found that if we require that a surface wave to decay away from the surface with a real exponent, the equation for the speed of the surface wave leads to a complex value of c_R, which means that the surface waves are leaky, and therefore cannot be accepted as a

stationary solution of the wave equation. However an experience with quasi longitudinal surface waves had shown that we can accept a complex exponent. Then we may expect to find a real value for c_R.

A surface wave (the Rayleigh one) satisfies the same equation (78) as any other wave, but with $\Upsilon = (c_R/c_t)^2$, which we earlier (see Eq. (58)) denoted as x^2. With it we rewrite Eq. (78) as

$$\left[x^2 - 1 - q^2 + \xi(\tilde{\boldsymbol{k}}\cdot\boldsymbol{a})^2\right]\boldsymbol{A} = E\tilde{\boldsymbol{k}}(\tilde{\boldsymbol{k}}\cdot\boldsymbol{A}) -$$
$$-\xi\left(\boldsymbol{a}[(1+q^2)(\boldsymbol{a}\cdot\boldsymbol{A}) + (\tilde{\boldsymbol{k}}\cdot\boldsymbol{a})(\tilde{\boldsymbol{k}}\cdot\boldsymbol{A})] + \tilde{\boldsymbol{k}}(\tilde{\boldsymbol{k}}\cdot\boldsymbol{a})(\boldsymbol{a}\cdot\boldsymbol{A})]\right). \tag{80}$$

The easiest way is to find x by try and error method. We suggest some value of $x = x_1 < 1$, seek the solution of (80) in the form (79). Multiply both parts by $\boldsymbol{n}$ and $\boldsymbol{\tau}$, obtain two homogeneous linear equations for α and β, find its determinant, which is a polynomial of 4-th order in powers of q: $D_4(q)$, and find its roots q_i (i=1-4). If all the roots are complex, we choose two of them with negative imaginary parts, say q_2 and q_3. For them we find $\alpha^{(2,3)}$, $\beta^{(2,3)}$ and $\boldsymbol{A}^{(2,3)}(q_{2,3})$. After that we use (30) and obtain $\boldsymbol{B}^{(2,3)}$. With these vectors we construct a linear combination, which satisfies boundary conditions

$$\gamma\boldsymbol{B}^{(2)}(q_2) + \delta\boldsymbol{B}^{(3)}(q_3) = 0. \tag{81}$$

Multiplication of this equation by $\boldsymbol{n}$ and $\boldsymbol{\tau}$ gives us again a system of two equations. It is resolvable, if its determinant $D(x_1, q_2, q_3)$ is equal to zero.

For an arbitrary chosen x_1 the determinant $D(x_1, q_2, q_3) \neq 0$. Instead it is a complex number, say $D(x_1, q_2, q_3) = y_1 + iz_1$. Then we try another x_2 till we find $D(x_2, q_2, q_3) = -y_2 - iz_2$, where signs of $y_{1,2}$ and respectively of $z_{1,2}$ are the same. After that by narrowing the interval x_1, x_2 we find the limiting point x_0, where $D(x_0, q_2, q_3) = 0$. The Rayleigh speed is $c_R = x_0 c_t$. In the case of $E = 1.5$, $\xi = 0.4$ and $\cos\theta_a = 0.4$ we got $c_R = 0.6066c_t$.

V. CONCLUSION

We formulated the theory of elastic waves in isotropic media with the help of complex vector wave functions like in particle physics. We considered reflection and refraction of waves at an interface with mode conversion or in other words with double splitting of the reflected and refracted waves. We had shown that in the case of a transverse incident wave there is a critical grazing angle φ_c, below which the longitudinal reflected wave becomes of the surface type with a speed in the interval (c_t, c_l). The speed of the Rayleigh wave is a root of the equation $1/r = 0$ where r is one of reflection amplitudes.

The theory for isotropic media was generalized to anisotropic ones with a single vector of anisotropy and a specific term in the free energy of deformation. In such media the transverse and longitudinal waves become hybridization, reflection and refraction at an interface is accompanied in general by triple splitting of reflected and refracted waves, and all the reflected waves are nonspecular.

In some cases, when speeds of all the reflected waves are higher than that of the incident one, a plane wave at some critical grazing angle φ_{c1} can be expected to completely transform into a surface one, which violates the energy conservation law. Because of some counter intuitive properties of elastic waves in anisotropic media such a transformation does not take place. However there are two critical points in the grazing angle of the incident wave, which can be considered as a hint that a nonlinearity should come into play near these points. If the nonlinearity is included, then the phenomenon, like that one shown in fig. 5b), could be possible. Transformation of a plane wave into a surface one should lead to an exponential grows of the surface wave amplitude, which can be related to such natural phenomena as the devastating earth quakes.

In many other aspects the wave theory for anisotropic media is a alike to those for isotropic ones. It predicts the Rayleigh surface wave on a free surface and the Stonley wave on an interface. Theses surface waves have complex normal components of the wave vector, however it does not lead to violation of energy conservation, because the real part of this normal component does not create an energy flux from the surface. We would like to stress that the surface waves, which exponentially decay away from the interface and at the same time oscillate, are not so called "leaky surface waves", because their energy leaks nowhere. The leaky surface waves cannot exist as a stationary solution of the wave equation without introduction of some losses because of nonlinearity or scattering, otherwise they violate the law of energy conservation.

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